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A Method of computing
the common Logarithm.

by
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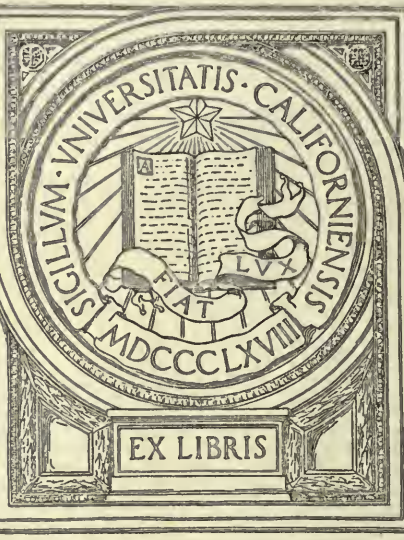

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A METHOD

OF

COMPUTING THE COMMON LOGARITHM

OF A NUMBER WITHOUT MAKING USE OF ANY LOGARITHM
BUT THAT OF SOME POWER OF 10,

By Artemas MARTIN (Washington).

The series usually given in text-books on Algebra for the computation of Logarithms contain, at least, the logarithm of one other number besides the one sought, and therefore necessitate the use of a Table.

It is the object of this paper to deduce from the ordinary logarithmic series, converging series for the computation of the common logarithm of any given number independent of any other logarithm but the logarithm of 10^m, which is readily seen by inspection to be m.

I shall make use of the various known forms of logarithmic series without stopping to show how they were deduced.

I. In the well-known series

$$\log_e(a+x) = \log_e a + \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 - \frac{1}{4}\left(\frac{x}{a}\right)^4 + \ldots,$$

$$\log_e(a-x) = \log_e a - \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{3}\left(\frac{x}{a}\right)^3 - \frac{1}{4}\left(\frac{x}{a}\right)^4 - \ldots.$$

put $a = 10^m$; then will

$$(1)\quad \log_{10}(10^m + x) = m + \mathrm{M}\left[\frac{x}{10^m} - \frac{1}{2}\left(\frac{x}{10^m}\right)^2 + \frac{1}{3}\left(\frac{x}{10^m}\right)^3 - \frac{1}{4}\left(\frac{x}{10^m}\right)^4 + \ldots\right],$$

$$(2)\quad \log_{10}(10^m - x) = m - \mathrm{M}\left[\frac{x}{10^m} + \frac{1}{2}\left(\frac{x}{10^m}\right)^2 + \frac{1}{3}\left(\frac{x}{10^m}\right)^3 + \frac{1}{4}\left(\frac{x}{10^m}\right)^4 + \ldots\right],$$

where $\mathrm{M} = 0,43429448190325182765 +$, the modulus of common or Brigg's logarithms.

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The series (1) and (2) are very convenient for the purposes of computation when x is a small number.

Examples. — 1. Find the common logarithm of 1002.

Solution. — Here $m = 3$ and $x = 2$, and (1) gives

$$\log_{10} 1002 = 3 + \mathrm{M}\left[\frac{2}{10^3} - \frac{1}{2}\left(\frac{2}{10^3}\right)^2 + \frac{1}{3}\left(\frac{2}{10^3}\right)^3 - \frac{1}{4}\left(\frac{2}{10^2}\right)^4 + \frac{1}{5}\left(\frac{2}{10^3}\right)^5 - \frac{1}{6}\left(\frac{2}{10^3}\right)^6 + \dots\right],$$
$$= 3 + \mathrm{M}(0,00199800266267 +),$$
$$= 3,000867721531 +,$$

which agrees to the last figure with the value given by Brigg's in his *Arithmetica Logarithmica* (London, 1624).

2. Find the common logarithm of 995.

Solution. — Here $m = 3$, $x = 5$, using (2), which gives

$$\log_{10} 995 = 3 - \mathrm{M}\left[\frac{5}{10^3} + \frac{1}{2}\left(\frac{5}{10^3}\right)^2 + \frac{1}{3}\left(\frac{5}{10^3}\right)^3 + \frac{1}{4}\left(\frac{5}{10^3}\right)^4 + \frac{1}{5}\left(\frac{5}{10^3}\right)^5 + \dots\right],$$
$$= 3 - \mathrm{M}(0,00501254182291 +),$$
$$= 3 - 0,00217691925 4 +,$$
$$= 2,997823080745 +,$$

which also agrees with Brigg's result.

II. Put $\mathrm{N} = 10^m(1 + x)$, then

$$(3) \qquad \begin{cases} \log_{10} \mathrm{N} = m + \mathrm{M} \log_e(1 + x), \\ \qquad = m + \mathrm{M}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \dots\right). \end{cases}$$

Put $\mathrm{N} = 10^m(1 - x)$, then

$$(4) \qquad \log_{10} \mathrm{N} = m - \mathrm{M}\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \dots\right).$$

When x is small, (3) and (4) will converge rapidly.

Examples. — 3. Find the common logarithm of 9900.

Solution. — $9900 = 10000\left(1 - \frac{1}{100}\right) = 10^4\left(1 - \frac{1}{100}\right)$; therefore

$m = 4$, $x = \frac{1}{100}$, and we have by (4)

$$\log_{10} 9900 = 4 - M\left[\left(\frac{1}{100}\right) + \frac{1}{2}\left(\frac{1}{100}\right)^2 + \frac{1}{3}\left(\frac{1}{100}\right)^3 + \frac{1}{4}\left(\frac{1}{100}\right)^4 + \frac{1}{5}\left(\frac{1}{100}\right)^5 + \frac{1}{6}\left(\frac{1}{100}\right)^6 + \frac{1}{7}\left(\frac{1}{100}\right)^7 + \dots\right],$$
$$= 3{,}99563519\,4597 +.$$

4. To find the common logarithm of 100160.

Solution. — $100160 = 100000\left(1 + \frac{16}{10000}\right) = 10^5\left(1 + \frac{16}{10000}\right)$; therefore $m = 5$, $x = \frac{16}{10000}$, and by (3) we have

$$\log_{10} 100160 = 5 + M\left[\left(\frac{16}{10000}\right) - \frac{1}{2}\left(\frac{16}{10000}\right)^2 + \frac{1}{3}\left(\frac{16}{10000}\right)^3 - \frac{1}{4}\left(\frac{16}{10000}\right)^4 + \frac{1}{5}\left(\frac{16}{10000}\right)^5 - \frac{1}{6}\left(\frac{16}{10000}\right)^6 + \dots\right].$$
$$= 5{,}00069\,43158\,66 +.$$

III. In the well-known series

$$\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \dots,$$

put $x = \sqrt[n]{a}$, and we have

$$(5)\quad \left\{ \log_{10} a = M\,n\left[(\sqrt[n]{a}-1) - \frac{1}{2}(\sqrt[n]{a}-1)^2 + \frac{1}{3}(\sqrt[n]{a}-1)^3 - \frac{1}{4}(\sqrt[n]{a}-1) + \frac{1}{5}(\sqrt[n]{a}-1) - \frac{1}{6}(\sqrt[n]{a}-1)^6 + \dots\right],\right.$$

where n may be any number whatever, and can be so chosen that $(\sqrt[n]{a}-1)$ will be a very small quantity, a being greater than unity, and then the series will converge rapidly, and only a few terms will be required to determine the logarithm sought to several places of decimals; but when a is a large number, the labor will be very great. It will be most advantageous to take n equal some power of 2.

If n be taken a very large number, one term of the series will be sufficient and in that case we will have.

$$(6)\qquad \log_{10} a = M\,n(\sqrt[n]{a}-1).$$

If greater accuracy be desired, take two terms and have

$$(7)\qquad \log_{10} a = M\,n\left[(\sqrt[n]{a}-1) - \frac{1}{2}(\sqrt[n]{a}-1)^2\right].$$

See Stephen Chase's *Treatise on Algebra,* New York (1849), pp. 252 and 254, where he takes $n = 2^{60}$. Brigg's took $n = 2^{54}$. A much smaller value of n will suffice when the logarithm is wanted to only 8 or 10 places of decimals.

Example. — 5. To find the common logarithm of 3.

Solution. — Taking $n = 2^{60}$, we have

$$\log_{10} 3 = \mathrm{M} \times n \times \left(\sqrt[n]{3} - 1\right) = 0,477121254719\,66+.$$

IV. In (5), put $a = \dfrac{b}{10^m}$; then we have

$$(8)\quad \left\{ \begin{aligned} \log_{10} b = m + \mathrm{M}n \Bigg\{ & \left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right] - \frac{1}{2}\left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right]^2 \\ & + \frac{1}{3}\left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right]^3 - \frac{1}{4}\left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right]^4 \\ & + \frac{1}{5}\left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right]^5 - \dots \Bigg\}, \end{aligned} \right.$$

where, as in (5), n may be any number chosen at pleasure, but it will be best to take n some power of 2. This is a convenient series for numbers greater than 10.

If n be a large number, one term of the series within the braces will give the logarithm with sufficient accuracy for ordinary purposes. In that case we have

$$(9)\qquad \log_{10} b = m + \mathrm{M}n\left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right].$$

If greater accuracy be required, take two terms and have

$$(10)\qquad \log_{10} b = m + \mathrm{M}n\left\{\left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right] - \frac{1}{2}\left[\sqrt[n]{\left(\frac{b}{10^m}\right)} - 1\right]^2\right\}.$$

V. $$\mathrm{Log}_e x = \left(1 - \frac{1}{x}\right) + \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 + \frac{1}{3}\left(1 - \frac{1}{x}\right)^3 + \frac{1}{4}\left(1 - \frac{1}{x}\right)^4 + \dots.$$

Substituting $\sqrt[n]{\left(\dfrac{b}{10^m}\right)}$ for x we have

$$(11)\quad \left\{ \begin{aligned} \log_{10} b = m + \mathrm{M}n \Bigg\{ & \left[1 - \sqrt[n]{\left(\frac{10^m}{b}\right)}\right] + \frac{1}{2}\left[1 - \sqrt[n]{\left(\frac{10^m}{b}\right)}\right]^2 \\ & + \frac{1}{3}\left[1 - \sqrt[n]{\left(\frac{10^m}{b}\right)}\right]^3 + \frac{1}{4}\left[1 - \sqrt[n]{\left(\frac{10^m}{b}\right)}\right]^4 + \dots \Bigg\}. \end{aligned} \right.$$

where n may be as in (8)

VI. $$\operatorname{Log}_e\left(\frac{a+x}{a-x}\right)=2\left[\frac{x}{a}+\frac{1}{3}\left(\frac{x}{a}\right)^3+\frac{1}{5}\left(\frac{x}{a}\right)^5+\frac{1}{7}\left(\frac{x}{a}\right)^7+\dots\right].$$

Put

$$x=\frac{a^2}{2y+a},$$

then

$$\frac{a+x}{a-x}=\frac{y+a}{y},$$

and

$$\log_e(y+a)=\log_e y+2\left[\frac{a}{2y+a}+\frac{1}{3}\left(\frac{a}{2y+a}\right)+\frac{1}{5}\left(\frac{a}{2y+a}\right)^5+\dots\right].$$

Let $y=10^m$, then we have

(12) $$\left\{\log_{10}(10^m+a)=m+2\operatorname{M}\left\{\frac{a}{2(10^m)+a}+\frac{1}{3}\left[\frac{a}{2(10^m)+a}\right]^3+\frac{1}{5}\left[\frac{a}{2(10^m)+a}\right]^5+\frac{1}{7}\left[\frac{a}{2(10^m)+a}\right]^7+\dots\right\}.\right.$$

Example. — 6. Find the common logarithm of 2.

Solution. — $2^{10}=1024=1000+24=10^3+24$; therefore

$$\log_{10}2^{10}=10\log_{10}2=\log_{10}(10^3+24),$$

and

$$\log_{10}2=\frac{1}{10}\log_{10}(10^3+24).$$

By (12),

$$\log_{10}2=\frac{3}{10}+\frac{2}{10}\operatorname{M}\left[\frac{3}{253}+\frac{1}{3}\left(\frac{3}{253}\right)^3+\frac{1}{5}\left(\frac{3}{253}\right)^5+\frac{1}{7}\left(\frac{3}{253}\right)^7+\frac{1}{9}\left(\frac{9}{253}\right)^9+\dots\right],$$

$$=0,30102999566\,3+.$$

If we put $x=\frac{a^2}{2y-a}$, we get $\frac{a+x}{a-x}=\frac{y}{y-a}$ and then

$$\log_e(y-a)=\log_e y-2\left[\frac{a}{2y-a}+\frac{1}{3}\left(\frac{a}{2y-a}\right)^3+\frac{1}{5}\left(\frac{a}{2y-a}\right)^5+\dots\right].$$

Take as before $y=10^m$ and we have

(13) $$\left\{\log_{10}(10^m-a)=m-2\operatorname{M}\left\{\frac{a}{2(10^m)-a}+\frac{1}{3}\left[\frac{a}{2(10^m)-a}\right]^3+\frac{1}{5}\left[\frac{a}{2(10^m)-a}\right]^5+\frac{1}{7}\left[\frac{a}{2(10^m)-a}\right]^7+\dots\right\}.\right.$$

Examples. — 7. Find the common logarithm of 3.

Solution. — $\operatorname{Log}_{10} 3 = \frac{1}{2}\log_{10} 9 = \frac{1}{2}\log_{10}(10-1)$; therefore by (13),

$$\log_{10} 3 = 1 - 2M\left[\frac{1}{19} + \frac{1}{3}\left(\frac{1}{19}\right)^3 + \frac{1}{5}\left(\frac{1}{19}\right)^5 + \frac{1}{7}\left(\frac{1}{19}\right)^7 + \frac{1}{9}\left(\frac{1}{19}\right)^9 + \cdots\right],$$
$$= 0,477121254719 + .$$

8. To find the common logarithm of 5.

Solution. — $\log_{10} 5 = \frac{1}{3}\log_{10} 125 = \frac{1}{3}\log_{10}(100+25)$; therefore, by (12),

$$\log_{10} 5 = \frac{2}{3} + 2M\left[\frac{1}{9} + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \frac{1}{5}\left(\frac{1}{9}\right)^5 + \frac{1}{7}\left(\frac{1}{9}\right)^7 + \frac{1}{9}\left(\frac{1}{9}\right)^9 + \cdots\right],$$
$$= 0,698970004336 + .$$

9. To find the common logarithm of 7.

Solution. — $\operatorname{Log}_{10} 7 = \log_{10}(10-3)$; therefore, by (13),

$$\log_{10} 7 = 1 - 2M\left[\frac{3}{17} + \frac{1}{3}\left(\frac{3}{17}\right)^3 + \frac{1}{5}\left(\frac{3}{17}\right)^5 + \frac{1}{7}\left(\frac{3}{17}\right)^7 + \frac{1}{9}\left(\frac{3}{19}\right)^9 + \cdots\right],$$
$$= 0,845098040014 + .$$

10. To find the common logarithm of 11.

Solution. — $\operatorname{Log}_{10} 11 = \log_{10}(10+1)$; therefore, by (12),

$$\log_{10} 11 = 1 + 2M\left[\frac{1}{21} + \frac{1}{3}\left(\frac{1}{21}\right)^3 + \frac{1}{5}\left(\frac{1}{21}\right)^5 + \frac{1}{7}\left(\frac{1}{21}\right)^7 + \frac{1}{9}\left(\frac{1}{21}\right)^9 + \cdots\right],$$
$$= 1,041392685158 + .$$

VII. — Take

$$N = 10^m\left(1+\frac{1}{10}\right)^{m_1}\left[1+\frac{1}{(10)^2}\right]^{m_2}\left[1+\frac{1}{(10)^3}\right]^{m_3}\left[1+\frac{1}{(10)^4}\right]^{m_4}\cdots;$$

then

$$(14)\quad \begin{cases} \log_{10} N = m + m_1\log_{10}\left(1+\frac{1}{10}\right) + m_2\log\left[1+\frac{1}{(10)^2}\right] + m_3\log_{10}\left[1+\frac{1}{(10)^3}\right] \\ \qquad + m_4\log_{10}\left[1+\frac{1}{(10)^4}\right] + m_5\log_{10}\left[1+\frac{1}{(10)^5}\right] + \cdots . \end{cases}$$

The logarithm of $\left[1+\frac{1}{(10)^n}\right]$ may be had from the series

$$\log_{10}(1+x) = \mathrm{M}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{6}x^6+\ldots\right),$$

by putting $x=\frac{1}{(10)^n}$; then

$$\log_{10}\left[1+\frac{1}{(10)^n}\right] = \mathrm{M}\left[\frac{1}{(10)^n}-\frac{1}{2(10^n)^2}+\frac{1}{3(10^n)_3}-\frac{1}{4(10^n)^4}+\ldots\right].$$

The above method is a modification of that made use of by Oliver Byrne in his little book on logarithms, but it requires the use of Tables or the computation of other logarithms than the one sought.

A RIGOROUS METHOD

OF

FINDING BIQUADRATE NUMBERS

WHOSE SUM IS A BIQUADRATE,

By Artemas MARTIN (Washington).

Euler and others have shown that the sum of two biquadrate numbers can not be a biquadrate; and Euler asserts (*Commentationes Arithmeticæ Collectæ*, vol. II, p. 281-282, 450, 456) that $a^4 + b^4 + c^4 = d^4$ is impossible, and states that he had not been able to find four biquadrate numbers whose sum is a biquadrate number, but that he had noticed that it is possible to determine in many ways *five* biquadrates whose sum is a biquadrate. The writer has not been able to find that Euler actually obtained any sets of five biquadrate numbers whose sum is a biquadrate.

In the *Mathematical Magazine*, vol. II, n° 10 (January, 1896), the writer published a paper, *About biquadrate numbers whose sum is a biquadrate,* in which he employed tentative processes to find many sets of biquadrate numbers whose sum is a biquadrate.

I. To find five biquadrate numbers whose sum is a biquadrate number.

Solution. — Let v, w, x, y, z be the roots of the required biquadrates, and s the root of their sum; then

$$v^4 + w^4 + x^4 + y^4 + z^4 = s^4. \tag{1}$$

Assume $v = 2a$, $w = a - b$, $x = a + b$, $y = d$,

$$z = \frac{d^2 - e^2}{2e} \quad \text{and} \quad s = \frac{d^2 + e^2}{2e};$$

then, by substitution,

$$(2a)^4 + (a-b)^4 + (a+b)^4 + d^4 + \left(\frac{d^2 - e^2}{2e}\right)^4 = \left(\frac{d^2 + e^2}{2e}\right)^4.$$

Expanding, transposing, and uniting terms,

$$2(3a^2+b^2)^2+d^4=\frac{d^2(d^4+e^4)}{2e^2}.$$

Transposing d^4 and multiplying by $2e^2$,

$$4e^2(3a^2+b^2)^2=d^2(d^4+e^4)-2d^4e^2=d^2(d^4-2d^2e^2+e^4)=d^2(d^2-e^2)^2.$$

Extracting square root,

$$2e(3a^2+b^2)=d(d^2-e^2);$$

whence

$$b^2=\frac{d(d^2-e^2)}{2e}-3a^2. \tag{2}$$

In (2), take $d=2e$ and we get

$$b^2=3e^2-3a^2=3(e^2-a^2)=3(e+a)(e-a), \qquad =\frac{m^2}{n^2}(e-a)^2 \text{ say};$$

whence

$$a=\frac{(m^2-3n^2)e}{m^2+3n^2}.$$

Taking $e=2(m^2+3m^2)$ we have

$$a=2(m^2-3n^2), \qquad b=\frac{m}{n}(e-a)=12mn, \qquad d=2e=4(m^2+3n^2);$$

therefore

$$v=2a=4(m^2-3n^2), \qquad w=a-b=2m^2-12mn-6n^2,$$
$$x=a+b=2m^2+12mn-6n^2, \qquad y=d=2e=4(m^2+3n^2),$$
$$z=\frac{d^2-e^2}{2e}=\frac{3e}{2}=3(m^2+3n^2), \qquad s=\frac{d^2+e^2}{2e}=\frac{5e}{2}=5(m^2+3n^2).$$

Substituting in (1) we have

$$\left\{\begin{array}{l}(4m^2-12n^2)^4+(2m^2-12mn-6n^2)^4+(2m^2+12mn-6n^2)^4\\ \qquad\qquad +(4m^2+12n^2)^4+(3m^2+9n^2)^4=(5m^2+15n^2)^4.\end{array}\right. \tag{3}$$

Examples. — 1. Taking $m=1$, $n=1$, we have, after dividing by 4^4

$$2^4+2^4+3^4+4^4+4^4=5^4.$$

2. Taking $m=2$, $n=1$, we have

$$4^4+21^4+22^4+26^4+28^4=35^4.$$

3. Taking $m=1$, $n=2$, we have

$$2^4+39^4+44^4+46^4+52^4=65^4.$$

4. Taking $m = 4$, $n = 1$, we have

$$22^4 + 52^4 + 57^4 + 74^4 = 76^4 = 95^4.$$

5. Taking $m = 1$, $n = 4$, we have

$$46^4 + 142^4 + 147^4 + 188^4 + 196^4 = 245^4.$$

6. Taking $m = 4$, $n = 3$, we have

$$44^4 + 122^4 + 129^4 + 166^4 + 172^4 = 215^4.$$

7. Taking $m = 5$, $n = 2$, we have

$$52^4 + 94^4 + 111^4 + 146^4 + 148^4 = 185^4.$$

8. Taking $m = 2$, $n = 5$, we have

$$22^4 + 237^4 + 262^4 + 284^4 + 316^4 = 395^4.$$

9. Taking $m = 2$, $n = 3$, we have

$$26^4 + 92^4 + 93^4 + 118^4 + 124^4 = 155^4.$$

10. Taking $m = 1$, $n = 6$, we get

$$142^4 + 286^4 + 327^4 + 428^4 + 436^4 = 545^4.$$

11. Taking $m = 7$, $n = 2$, we get

$$94^4 + 148^4 + 183^4 + 242^4 + 244^4 = 305^4.$$

12. Taking $m = 2$, $n = 7$, we get

$$118^4 + 453^4 + 454^4 + 572^4 + 604^4 = 755^4.$$

13. Taking $m = 7$, $n = 4$, we get

$$4^4 + 291^4 + 334^4 + 338^4 + 388^4 = 485^4.$$

14. Taking $m = 4$, $n = 7$, we get

$$74^4 + 489^4 + 524^4 + 598^4 + 652^4 = 815^4.$$

15. Taking $m = 8$, $n = 1$, we get

$$26^4 + 201^4 + 218^4 + 244^4 + 268^4 = 335^4.$$

16. Taking $m = 1$, $n = 8$, we get

$$286^4 + 478^4 + 579^4 + 764^4 + 772^4 = 965^4.$$

When $n = 1$ in (3), then

$$(4)\qquad \begin{cases} (4m^2-12)^4+(2m^2-12m-6)^4+(2m^2+12m-6)^4 \\ \qquad +(4m^2+12)^4+(3m^2+9)^4=(5m^2+15)^4, \end{cases}$$

and when $m = 1$ in (3), then

$$(5)\qquad \begin{cases} (12n^2-4)^4+(6n^2+12n-2)^4+(6n^2-12n-2)^4 \\ \qquad +(12n^2+4)^4+(9n^2+3)^4=(15n^2+5)^4. \end{cases}$$

Examples. — 17. Taking $m = 10$ in (4), we get

$$74^4+309^4+314^4+388^4+412^4=515^4.$$

18. Taking $n = 11$ in (5), we get

$$149^4+214^4+273^4+362^4+364^4=455^4.$$

Take $d = 2$ in (2), then

$$b^2=\frac{4-e^2}{e}-3a^2=\frac{4-e^2-3a^2e}{e}.$$

Put $e = t^2$; then

$$b^2=\frac{4-t^4-3a^2t^2}{t^2} \qquad \text{and} \qquad b=\frac{\sqrt{(4-3a^2t^2-t^4)}}{t}.$$

Now let

$$4-3a^2t^2-t^4=(2-3t^2)^2$$

and we get

$$3a^2=12-10t^2, \qquad \text{or} \qquad a^2=4-\frac{10}{3}t^2=\left(2-\frac{p}{q}t\right)^2;$$

whence

$$t=\frac{12pq}{3p^2+10q^2}.$$

$$a=2-\frac{p}{q}t=\frac{20q^2-6p^2}{3p^2+10q^2}, \qquad b=\frac{2-3t^2}{t},$$

$$e=\quad t^2\quad=\left(\frac{12pq}{3p^2+10q^2}\right)^2, \qquad d=2.$$

Another Method. — Assume

$$(6)\qquad (2x)^4+(x+y)^4+(x-y)^4+w^4+z^4=s^4.$$

Expanding and uniting terms we have

$$2(3x^2+y^2)^2+w^4+z^4=s^4.$$

Now take

$$3x^2+y^2=wz$$

and we have

$$w^2 + z^2 = s^2,$$

which is satisfied by

$$w = p^2 - q^2, \qquad z = 2pq, \qquad s = p^2 + q^2;$$

and by substitution (6) becomes

$$(7) \qquad (2x)^4 + (x+y)^4 + (x-y)^4 + (p^2-q^2)^4 + (2pq)^4 = (p^2+q^2)^4.$$

But

$$y^2 + 3x^2 = wz = 2pq(p^2 - q^2),$$

whence

$$y = \pm\sqrt{[2pq(p^2-q^2) - 3x^2]}.$$

Hence we must have

$$(8) \qquad 2pq(p^2-q^2) - 3x^2 = \square.$$

Let $p = 2$, $q = 1$, then (8) becomes

$$12 - 3x^2 = 3(2-x)(2+x) = \square = \frac{m^2}{n^2}(2-x)^2 \text{ say,}$$

and we get by reduction

$$x = \frac{2m^2 - 6n^2}{m^2 + 3n^2},$$

$$y = \frac{m}{n}(2-x) = \frac{12mn}{m^2+3n^2}, \qquad 2x = \frac{4m^2 - 12n^2}{m^2+3n^2},$$

$$x+y = \frac{2m^2 + 12mn - 6n^2}{m^2+3n^2}, \qquad x-y = \frac{2m^2 - 12mn - 6n^2}{m^2+3n^2},$$

$$w = 3, \qquad z = 4, \qquad s = 5.$$

Substituting these values in (6), and multiplying by $(m^2+3n^2)^4$, we have (3).

The artifice of putting $3x^2 + y^2 = wz$ is due to Mr. Cyrus-B. Haldeman of Ross, Butler Co, Ohio (U. S. A.), who has contributed an extended paper on biquadrates which will be published in n° 12, vol. II, of the *Mathematical Magazine* edited and published by the writer.

In order to obtain a more general solution of

$$2pq(p^2-q^2) - 3x^2 = \square,$$

put

$$2pq(p^2-q^2) = 3b^2 = \frac{3m^2}{n^2}(p+q)^2(p-q)^2;$$

then

$$\frac{3m^2}{n^2}(p+q)^2(p-q)^2-3x^2=\square,$$

or

$$3\left[\frac{m}{n}(p+q)(p-q)+x\right]\times\left[\frac{m}{n}(p+q)(p-q)-x\right]=\square.$$

Put

$$3\left[\frac{m}{n}(p+q)(p-q)+x\right]$$
$$\times\left[\frac{m}{n}(p+q)(p-q)-x\right]=\frac{r^2}{s^2}\left[\frac{m}{n}(p+q)(p-q)-x\right]^2;$$

then

$$3\left[\frac{m}{n}(p+q)(p-q)+x\right]=\frac{r^2}{s^2}\left[\frac{m}{n}(p+q)(p-q)-x\right],$$

from which we get

$$x=\frac{(mr^2-3ns^2)(p^2-q^2)}{n(r^2+3s^2)}.$$

From

$$2pq=\frac{3m^2}{n^2}(p^2-q^2)=\frac{3m^2p^2-3m^2q^2}{n^2}$$

we get

$$9p^2-\frac{6n^2pq}{m^2}=9q^2,$$

whence

$$\frac{p}{q}=\frac{n^2\pm\sqrt{(9m^4+n^4)}}{3m^2}.$$

We must yet make

$$9m^4+n^4=\square. \tag{9}$$

This is satisfied by $n=2m$; let, therefore,

$$m=t+b,\qquad n=2t+c,$$

and by substitution and reduction we have

$$25t^4+(36b+32c)t^3+(54b^2+24c^2)t^2+(36b^3+8c^3)t+9b^4+c^4=\square. \tag{10}$$

Assume

$$(10)=\left[5t^2+\left(\frac{18b+16c}{5}\right)t+A\right]^2;$$

then by substitution and equating like terms we get

$$10A + \frac{1}{25}(18b + 16c)^2 = 54b^2 + 24c^2, \tag{11}$$

$$\frac{2}{5}A(18b + 16c)t + A^2 = (36b^3 + 8c^3)t + 9b^4 + c^4. \tag{12}$$

From (11) and (12) we easily find

$$A = \frac{25(54b^2 + 24c^2) - (18b + 14c)^2}{250} = \frac{513b^2 - 288bc + 172c^2}{125},$$

$$t = \frac{5(9b^4 + c^4 - A^2)}{2A(18b + 16c) - 5(36b^3 + 8c^3)}.$$

Retracing our steps we have

$$t + b = m, \qquad 2t + c = n;$$

$$\frac{p}{q} = \frac{n^2 \pm \sqrt{\left[5t^2 + \frac{1}{5}(18b + 16c)t + A\right]}}{3m^2},$$

$$x = \frac{(mr^2 - 3ns^2)(p^2 - q^2)}{n(r^2 + 3s^2)}, \qquad y = \pm\frac{r}{s}\left[\frac{m}{n}(p^2 - q^2) - x\right],$$

$$w = p^2 - q^2, \qquad z = 2pq.$$

This result is to complicated for convenient use, and the numbers in most cases, if not in all, will be large. Special solutions may be obtained which whill give better results.

If in (9) we put $n = 2m - d$ we get

$$25m^4 - 32m^3d + 24m^2d^2 - 8md^3 + d^4 = \square. \tag{13}$$

Now put

$$(13) = (5m^2 - 4md + d^2)^2$$

and we find

$$d = -4m, \qquad n = -2m,$$

and then

$$\frac{p}{q} = \frac{3}{1}.$$

Hence we may take $p = 3, q = 1$, which values being substituted in (8) lead us to the same result as arrived at in (3).

Putting $3b^2$ for $2pq(p^2 - q^2)$ in (8) we have

$$3(b^2 - x^2) = 3(b + x)(b - x) = \square, \qquad = \frac{m^2}{n^2}(b - x)^2 \text{ say},$$

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whence

$$x = \frac{b(m^2 - 3n^2)}{m^2 + 3n^2};$$

then

$$y = \frac{m}{n}(b - x) = \frac{6bmn}{m^2 + 3n^2},$$

$$x + y = \frac{b(m^2 + 6mn - 3n^2)}{m^2 + 3n^2}, \qquad x - y = \frac{b(m^2 - 6mn - 3n^2)}{m^2 + 3n^2}.$$

Therefore we have by (7)

$$(14) \quad \left[\frac{2b(m^2 - 3n^2)}{m^2 + 3n^2}\right]^4 + \left[\frac{b(m^2 - 6mn - 3n^2)}{m^2 + 3n^2}\right]^4 + \left[\frac{b(m^2 + 6mn - 3n^2)}{m^2 + 3n^2}\right]^4 + (p^2 - q^2)^4 + (2pq)^4 = (p^2 + q^2)^4.$$

For another solution of (8), put

$$p = s + 2q \qquad \text{and} \qquad x = t + 2q^2;$$

then by substitution we get

$$(15) \quad 2qs^3 + 12q^2s^2 + 22q^3s - 3t^2 - 12tq^2 = \square.$$

Taking

$$-3t^2 - 12tq^2 = 0,$$

we have

$$t = -4q,$$

and then

$$2qs^3 + 12q^2s^2 + 22q^3s = \square, \qquad = \frac{4m^2}{n^2}q^2s^2 \text{ say};$$

whence

$$s^2 + 6qs + 11q^2 = \frac{2m^2}{n^2}qs.$$

By transposition,

$$s^2 + \left(6 - \frac{2m^4}{n^2}\right)qs = -11q^2;$$

whence

$$s = \frac{q(m^2 - 3n^2) \pm q\sqrt{(m^4 - 6m^2n^2 - 2n^4)}}{n^2}.$$

$m = 3n$ satisfies $m^4 - 6m^2n^2 - 2n^2 = \square$,

and gives

$$s = 11, \quad q = 1; \quad \text{or} \quad s = 1, \quad q = 1;$$

and then

$$p = 13 \text{ or } 3;$$

also,

$$x = 2q^2 - 4q = -2;$$

$$y = \frac{2m}{n}(qs) = 3 \times 2 \times 11 = 66.$$

The values $p = 13$, $q = 3$ give, after dividing by 2^4,

$$2^4 + 13^4 + 32^4 + 34^4 + 84^4 = 85^4.$$

By similar methods six, seven, eight, and any number of biquadrates whose sum is a biquadrate may be found.

II. To find nine biquadrate numbers whose sum is a biquadrate number.

Solution. — Dividing (3) by $(m^2+n^2)^4$, and then multiplying the quotient by 5^4, and also by 3^4, we have

$$(16)\quad \left\{ \begin{aligned} &\left[\frac{20(m^2-3n^2)}{m^2+3n^2}\right]^4 + \left[\frac{10(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4 \\ &\quad + \left[\frac{10(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4 + 15^4 + 20^4 = 25^4, \end{aligned} \right.$$

and

$$(17)\quad \left\{ \begin{aligned} &\left[\frac{12(m^2-3n^2)}{m^2+3n^2}\right]^4 + \left[\frac{6(m^2-6mn+3n^2)}{m^2+3n^2}\right]^4 \\ &\quad + \left[\frac{6(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4 + 9^4 + 12^4 = 15. \end{aligned} \right.$$

Substituting the value of 15^4 from (17) in (16) we get

$$(18)\quad \left\{ \begin{aligned} &\left[\frac{12(m^2-3n^2)}{m^2+3n^2}\right]^4 + \left[\frac{6(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4 + \left[\frac{6(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4 \\ &+ \left[\frac{20(m^2-3n^2)}{m^2+3n^2)}\right]^4 + \left[\frac{10(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4 + \left[\frac{10(m^2+6mn+3n^2)}{m^2+3n^2}\right]^4 \\ &\qquad + 9^4 + 12^4 + 20^4 = 25^4. \end{aligned} \right.$$

Examples. — 19. If $m = 2, n = 1$, then $m^2 + 3n^2 = 7$ and we have

$$12^4 + 20^4 + 63^4 + 66^4 + 78^4 + 84^4 + 110^4 + 130^4 + 140^4 = 175^4.$$

20. If $m = 1$, $n = 2$, then $m^2 + 2n^2 = 13$ and we have

$$6^4 + 10^4 + 117^4 + 132^4 + 138^4 + 156^4 + 220^4 + 230^4 + 260^4 = 325^4.$$

III. To find thirteen biquadrate numbers whose sum is a biquadrate number.

Solution. — Multiplying (18) by 5^4,

$$(19)\quad \left\{\begin{aligned} &\left[\frac{60(m^2-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{30(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{30(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4\\ &+\left[\frac{100(m^2-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{50(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{50(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4\\ &\qquad\qquad +45^4+60^4+100^4=125^4. \end{aligned}\right.$$

Multiplying (17) by 3^4,

$$(20)\quad \left\{\begin{aligned} &\left[\frac{36(m^2-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{18(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4\\ &\qquad +\left[\frac{18(m^2+6m-3n^2)}{m^2+3n^2}\right]^4+27^4+36^4=45^4. \end{aligned}\right.$$

Putting the left-hand member of (20) for 45^4 in (19), we have

$$(21)\quad \left\{\begin{aligned} &\left[\frac{36(m^2-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{18(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{18(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4\\ &+\left[\frac{60(m^2-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{30(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{30(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4\\ &+\left[\frac{100(m^2-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{50(m^2-6mn-3n^2)}{m^2+3n^2}\right]^4+\left[\frac{50(m^2+6mn-3n^2)}{m^2+3n^2}\right]^4\\ &\qquad\qquad +27^4+36^4+60^4+100^4=125^4. \end{aligned}\right.$$

Example. — **21.** If $m=2$ and $n=1$, $m^2+3n^2=7$ and

$$36^4+60^4+100^4+189^4+198^4+234^4+252^4+330^4$$
$$+390^4+420^4+550^4+650^4+700^4=875^4.$$

In the same way can be found 17, 21, 25, 29, ..., $5+4t$ biquadrate numbers whose sum is a biquadrate number, where $t=1, 2, 3, 4, 5, \ldots$.

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